On the effect of Couple Stresses on the Magnetohydrodynamic (MHD) of a Non-Newtonian Unsteady Flow Between two Parallel Porous Plates

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Abstract
The study of magnetohydrodynamic (MHD) flow has received much attention in the past years owing to its application in MHD generations, plasma studies, unclear reactor, geothermal energy extraction purification of metal from non-metal enclosure, polymer technology and metallurgy. In oscillatory flow, problems with constant pressure gradient is to find either a numerical solution or analytical solution by reducing the partial differential equation to ordinary differentiation equation or system of equations. In this paper, the initial-boundary value problem that modelled the unsteady visco-elastic hydromagnetic fluid flow through a porous channel is revisited. A new method of solution is proposed in the steady and unsteady flow regimes. Computation of solution relied on the semi-numerical and rapidly convergent Adomian decomposition method (ADM). The convergence and the boundedness of the series solution are validated and presented in tabular form. Tabular and graphical results feature prominently in this work to explain the effect of pertinent flow parameters, such as magnetic field, couple-stress, velocity. From the results, it was observed that an increase in the couple stress parameter will ultimately decrease the velocity maximum as a result of fluid thickening. Also, it is observed that an increase in Hartman number decreases the flow maximum velocity due to the retarding effect of electromagnetic fields produced across the channel.

Keywords: Couple stresses, ADM, initial-boundary value problem, viscoelastic fluid

Introduction
In recent times, modelling of unsteady fluid flows has been on the increase due to its numerous applications in engineering, medicine, pharmaceutical industries to mention just a few. Of interest is the work of, Eldabe et al. (2003) in which the effect of couple stresses on magnetohydrodynamics of a Non-Newtonian unsteady flow of an incompressible fluid through porous channel has been studied by using the well-known Eyring-Powell model. The pulsatile case of this problem was investigated by Zuezo and Beg (2009) using the Network Simulation Method (NSM). While the heat transfer to the couple stress fluid flow was studied by Adesanya and Makinde (2014), other important studies involving unsteady flows in the purely Newtonian cases have been carried out by Adesanya and Makinde (2012), Mehmood and Ali (2007), Jha and Ajibade (2010) and many more.

The usual practice in oscillatory flow problems with constant pressure gradient is to find either a numerical solution or better still analytical solution by reducing the Partial differential equation to Ordinary differential equation or system of equation depending on either constant or pulsatile pressure gradient respectively. By so doing, the contribution of the initial condition is rendered redundant. In this paper, a semi-numerical or semi-analytical approach in the handling of initial-boundary value problem that will account for both boundary and initial conditions is presented, in such a way that, the general solution will then be the combination of the time-dependent solution.
To achieve this objective, a powerful Adomian decomposition method will be applied in obtaining a convergent series solution for the newly transformed initial-boundary problem and the steady boundary valued problem.

**Materials and Methods**

A classical example to explain the proposed method, we consider the well-posed linearized dimensionless equation for the visco-elastic couple stress model given by Eldabe et al. (2003) to be

\[
\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial y} = \frac{1}{\text{Re}} \left( \frac{1 + M}{M} \right) \frac{\partial^2 u}{\partial y^2} - \frac{Ha^2}{\text{Re}} u - \frac{1}{a^2 \text{Re} \frac{\partial^4 u}{\partial y^4}},
\]

subject to appropriate initial condition

\[u(0, y) = \sin(\pi y), \quad 0 < y < 1\] (1.2)

the no-slip condition

\[u(t, 0) = 0 = u(t, 1)\] (1.3)

and the stress free condition

\[u'(t, 0) = 0 = u'(t, 1)\] (1.4)

where the following dimensionless parameters and variables have been used

\[
u' = \frac{u}{v_0}, \quad x' = \frac{x}{h}, \quad y' = \frac{y}{h}, \quad t' = \frac{v_0 t}{h}, \quad \lambda = \frac{h^2}{dP} dx
\]

\[
p' = \frac{p}{\rho v_0^2}, \quad \beta = \frac{\alpha^2 h^2}{\mu_0 v}, \quad \text{Re} = \frac{v_0 h}{\eta}, \quad a^2 = \frac{\mu h^2}{\beta \mu C}, \quad M = \frac{1}{\beta \mu C}
\]

in order to obtain the solution of the non-homogeneous initial boundary value problem (1.1) - (1.3),

\[u(t, y) = v(y) + w(t, y)\] (1.6)

where \(v(y), w(t, y)\) represents the steady and unsteady solution respectively. Substituting (1.6) in (1.1) gives

\[
\frac{\partial w}{\partial t} + \frac{\partial w}{\partial y} + \frac{dv}{dy} = \frac{1}{\text{Re}} \left( \frac{1 + M}{M} \right) \frac{\partial^2 w}{\partial y^2} + \left( \frac{1 + M}{M} \right) \frac{d^2 v}{dy^2}
\]

\[\frac{Ha^2}{\text{Re}} w - \frac{Ha^2}{\text{Re} v} - \frac{1}{a^2 \text{Re} \frac{\partial^4 w}{\partial y^4}} - \frac{1}{a^2 \text{Re} \frac{d^4 v}{dy^4}}\] (1.7)
Using (1.6), the initial condition can be transformed as follows
\[ u(0, y) = v(y) + w(0, y) = \sin(\pi y) \]
\[ \Rightarrow w(0, y) = \sin(\pi y) - v(y) \]  
(1.8)

while the boundary condition gives
\[ u(t, 0) = v(0) + w(t, 0) = 0 \]
\[ u_t(t, 0) = v_t(0) + w_t(t, 0) = 0 \]
\[ u_t(t, 1) = v_t(1) + w_t(t, 1) = 0 \]
\[ u(t, 0) = v(0) + w(t, 0) = 0 \]  
(1.9)

Resolving (1.7) - (1.9) into steady and unsteady parts, we get
\[ \frac{d^4 v}{dy^4} = a^2 \left( \text{Re} \lambda + (1 + M) \frac{d^2 v}{dy^2} - H(1 + M) \frac{d^2 v}{dy^2} - \text{Re} \frac{dv}{dy} \right) \]  
(1.10)

subject to
\[ v(0) = v'(0) = v(1) = v''(1) = 0 \]  
(1.11)

while the unsteady part can be written as
\[ \frac{\partial w}{\partial t} + \frac{\partial w}{\partial y} = \left( \frac{1 + M}{\text{Re}} \right) \frac{\partial^2 w}{\partial y^2} - \frac{H a^2}{\text{Re}} \frac{w}{a^2} - \frac{1}{\text{Re} a^2} \frac{\partial^4 w}{\partial y^4} \]  
(1.12)

subject to
\[ w(0, y) = \sin(\pi y) - v(y) \]
\[ w(0) = w''(0) = w(1) = w''(1) = 0 \]  
(1.13)

Observe that the solution of \( w(t, y) \) depends on the solution of \( v(y) \).

By Adomian decomposition method, we assume an infinite series solution in the form
\[ v = \sum_{n=0}^{\infty} v_n(y) \]  
(1.14)

the integral form of (2.7) - (2.8) gives
\[ v(y) = \sum_{n=0}^{\infty} \left( \frac{d^4 v}{dy^4} \right) f_n dy + \sum_{n=0}^{\infty} \left( \frac{d^3 v}{dy^3} \right) f_n dy dy \]
\[ + \sum_{n=0}^{\infty} a^2 \left[ \text{Re} \lambda - \text{Re} \frac{dv}{dy} + (1 + M) \frac{d^2 v}{dy^2} - H (1 + M) \frac{d^2 v}{dy^2} \right] f_n dy dy dy \]  
(1.15)

substituting (1.11) in (1.13), one obtains
\[ \sum_{n=0}^{\infty} v_n(y) = \sum_{n=0}^{\infty} f_1 dy + \sum_{n=0}^{\infty} f_2 dy dy \]
\[ + \sum_{n=0}^{\infty} a^2 \left[ \text{Re} \lambda - \text{Re} \frac{d}{dy} \left( \sum_{n=0}^{\infty} v_n \right) + (1 + M) \frac{d^2}{dy^2} \left( \sum_{n=0}^{\infty} v_n \right) - H (1 + M) \frac{d^2}{dy^2} \left( \sum_{n=0}^{\infty} v_n \right) \right] dy dy dy dy \]  
(1.16)

where
\[ f_1 = \frac{d v(0)}{dy}, f_2 = \frac{d^3 v(0)}{dy^3} \]  
(1.17)

the recurrence relation can be written as
In this section, the convergence and the boundedness of the series solutions are validated and presented below as Table 1.

\[ v_0(y) = \int_0^y f_1 dy + \int_0^y \int_0^y f_2 \, dx \, dy + \int_0^y \int_0^y \int_0^y a^2 \, Re \, \frac{d^3}{dy^3} \, dy \, dy \, dy \]  

\[ v_{n+1}(y) = \int_0^y \int_0^y \int_0^y a^2 \left[ Re \, \lambda - Re \, \frac{d^2}{dy^2} v_n + (1 + M) \frac{d^2}{dy^2} v_n - H^2 v_n \right] \, dy \, dy \, dy \]  

(1.18)

Due to convergence, the approximate solution can be written as the truncated series

\[ v = \sum_{n=0}^N v_n(y) \]  

(1.19)

Similarly, integrating (1.12) with respect to time

\[ w(t, y) = w(0, y) + \int_0^t \left[ \left( \frac{1 + M}{Re} \right) \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial y} \frac{Ha^2}{Re} w - \frac{1}{a^2 \, Re \, \frac{\partial^4 w}{\partial y^4}} \right] \, dt \]  

(1.20)

Introducing the infinite series solution

\[ w = \sum_{n=0}^\infty w_n(t, y) \]  

(1.21)

Substituting (1.21) in (1.20), leads to

\[ \sum_{n=0}^\infty w_n(t, y) = w(0, y) + \int_0^t \left[ \left( \frac{1 + M}{Re} \right) \frac{\partial^2}{\partial y^2} \left( \sum_{n=0}^\infty w_n \right) - \frac{\partial}{\partial y} \left( \sum_{n=0}^\infty w_n \right) - \frac{Ha^2}{Re} \sum_{n=0}^\infty w_n - \frac{1}{a^2 \, Re \, \frac{\partial^4}{\partial y^4} \left( \sum_{n=0}^\infty w_n \right)} \right] \, dt \]  

(1.22)

So that the recurrence relation of (1.22) takes the form

\[ w_0(t, y) = \sin(\pi y) - v(y) \]  

\[ w_{n+1}(t, y) = \int_0^t \left[ \left( \frac{1 + M}{Re} \right) \frac{\partial^2 w_n}{\partial y^2} - \frac{\partial w_n}{\partial y} \frac{Ha^2}{Re} w_n - \frac{1}{a^2 \, Re \, \frac{\partial^4 w_n}{\partial y^4}} \right] \, dt \]  

(1.23)

In a similar manner, the approximate solution is the partial sum

\[ w = \sum_{n=0}^\infty w_n(t, y) \]  

(1.24)

Then the general solution of the initial-boundary value problem that is valid for all \( t \), is can be written as

\[ u(t, y) = \sum_{n=0}^\infty v_n(y) + \sum_{n=0}^\infty w_n(t, y) \]  

(1.25)

Particularly, the initial condition \( u(0, y) \) can be readily recovered fully from (1.25) in the form

\[ u(0, y) = \sum_{n=0}^\infty v_n(y) + \sin(\pi y) - \sum_{n=0}^\infty v_n(y) \]  

= \sin(\pi y)  

(1.26)

**Results**

In this section, the convergence and the boundedness of the series solutions are validated and presented below as Table 1.
Table 1: Convergence of Solution when $a = 1 = H = \text{Re}, M = 5$

<table>
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<tr>
<th>$(P, Q)$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$w(t, 1)$</th>
</tr>
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<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0.00414159</td>
</tr>
<tr>
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<td>-0.250096</td>
<td>0.00364592</td>
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<td>0.335394</td>
<td>0.00368286</td>
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<td>0.00368294</td>
</tr>
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<td>6</td>
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<td>-0.337085</td>
<td>0.00368294</td>
</tr>
</tbody>
</table>

While figures 1-5 represent the effect of variation of flow parameters, in figure 1, it is observed that an increase in the couple stress inverse has increasing effect on the fluid flow. Therefore, it is expected that an increase in the couple stress parameter will ultimately decrease the velocity maximum due to fluid thickening.
Figure 2: Effect of magnetic field intensity on the flow velocity

From figure 2, one observes that an increase in Hartman number decreases the flow maximum velocity due to the retarding effect of electromagnetic fields produced across the channel on the fluid flow. Figure 3 shows the effect of rise in the fluid elasticity. From the graph it is observed that an increase in the fluid elasticity parameter leads to decrease in the fluid velocity maximum within the channel. This is due to the fact that rise in fluid elasticity parameter means a decrease in the Eyring-Powel parameters. Therefore, the flow velocity increases as the Eyring-Powell parameters increase within the channel.

Figure 3: Effect of fluid viscosity on the flow velocity
Similarly, figure 4 represents the effect of fluid injection into the channel. It is observed that the Reynolds number increases as more and more fluid is injected into the channel there is corresponding increase in the flow velocity. However, flow remains laminar provided the critical Reynolds' number is not exceeded. Finally, figure 5 shows the oscillatory nature of the fluid flow within the channel.

**Conclusion**

The main goal of this work is to present a more general approach in the handling of initial-boundary problem arising from unsteady flow in fluid dynamics. To achieve this objective the semi-numerical Adomian decomposition method is applied to the partial differential equation without any need for a simplifying assumption, linearization, discretization, perturbation or initial guess.

**References**


